Chaipunya et al. *Fixed Point Theory and Applications* 2012, **2012**:219
http://www.fixedpointtheoryandapplications.com/content/2012/1/219

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

On $\mathcal{P}$-contractions in ordered metric spaces

Parin Chaipunya, Wutiphol Sintunavarat and Poom Kumam*

*Correspondence:
poom.kum@kmutt.ac.th
Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi, Bang Mod, Bangkok, 10140, Thailand

Abstract

In this paper, we introduced a new type of a contractive condition defined on an ordered space, namely a $\mathcal{P}$-contraction, which generalizes the weak contraction. We also proved some fixed point theorems for such a condition in ordered metric spaces. A supporting example of our results is provided in the last part of our paper as well.

MSC: 06A05; 06A06; 47H09; 47H10; 54H25

Keywords: $\mathcal{P}$-function; $\mathcal{P}$-contraction; metric space; partially ordered set

1 Introduction and preliminaries

It is well known that the Banach contraction principle has been improved in different directions in different spaces by mathematicians over the years. Even in the contemporary research, it remains a heavily investigated branch. Thus, several authors have generalized the principle in various ways (see, for example, [1–14]).

In 1997, Alber and Guerre-Delabriere [15] have introduced the concept of weak contraction in Hilbert spaces. Later, Rhoades [16] showed, in 2001, that these results are also valid in complete metric spaces. We state the result of Rhoades in the following.

A mapping $f : X \to X$, where (X, d) is a metric space, is said to be weakly contractive if

$$d(fx, fy) \le d(x, y) - \varphi\bigl(d(x, y)\bigr) \tag{1.1}$$

for all $x, y \in X$ and $\varphi : [0, +\infty) \to [0, +\infty)$ is a function satisfying:

(i) φ is continuous and nondecreasing;

(ii) $\varphi(t) = 0$ if and only if $t = 0$;

(iii) $\lim_{t\to+\infty} \varphi(t) = +\infty$.

Note that (1.1) reduces to an ordinary contraction when $\varphi(t) := kt$, where $0 \le k < 1$.

Theorem 1.1 ([16]) *Let (X, d) be a complete metric space and f be a weakly contractive mapping. Then f has a unique fixed point x^* in X.*

An interesting way to generalize this theorem is to consider it in case a partial ordering is defined on the space. Recall that a relation $\sqsubseteq$ is a partial ordering on a set X if it is reflexive, antisymmetric and transitive. By this meaning, we write $b \sqsupseteq a$ instead of $a \sqsubseteq b$ to emphasize some particular cases. Any $a, b \in X$ are said to be *comparable* if $a \sqsupseteq b$ or $a \sqsubseteq b$. If a set X has a partial ordering $\sqsubseteq$ defined on it, we say that it is a *partially ordered set* (*w.r.t.* $\sqsubseteq$) and denote it by $(X, \sqsubseteq)$. $(X, \sqsubseteq)$ is said to be a *totally ordered set* if any two elements in X are comparable. Moreover, it is said to be a *sequentially ordered set* if each element of a convergent sequence in X is comparable with its limit. Yet, if (X,d) is a metric space and $(X,\sqsubseteq)$ is a partially ordered (totally ordered, sequentially ordered) set, we say that X is a *partially ordered (totally ordered, sequentially ordered, respectively) metric space*, and it will be denoted by $(X,\sqsubseteq,d)$.

In 2009, Harjani and Sadarangani [17] carried the work of Rhoades [16] into partially ordered metric spaces. We now state the result proved in [17] as follows.

Theorem 1.2 ([17]) *Let $(X,\sqsubseteq,d)$ be a complete partially ordered metric space and let $f : X\to X$ be a continuous and nondecreasing mapping such that*

$$d(fx,fy)\le d(x,y)-\varphi\bigl(d(x,y)\bigr)$$

for $x\sqsubseteq y$, where $\varphi:[0,+\infty)\to[0,+\infty)$ is a function satisfying:

(i) *φ is continuous and nondecreasing*;

(ii) *$\varphi(t)=0$ if and only if $t=0$*;

(iii) *$\lim_{t\to+\infty}\varphi(t)=+\infty$.*

If there exists $x_0\in X$ such that $x_0\sqsubseteq fx_0$, then f has a fixed point.

Harjini and Sadarangani [17] also proved fixed point theorems for noncontinuous mappings, nonincreasing mappings and even for non-monotonic mappings.

The aim of this paper is to introduce a weak condition which resulted in the concept called a *$\mathcal{P}$-contraction*.

2 $\mathcal{P}$-functions

In this section, we introduce our concept of a $\mathcal{P}$-function and some of its fundamental properties. Not to be ambiguous, we assume that $\mathbb{R}$ represents the set of all real numbers while $\mathbb{N}$ represents the set of all positive integers.

Definition 2.1 Let $(X,\sqsubseteq,d)$ be a partially ordered metric space. A function $\varrho : X\times X\to\mathbb{R}$ is called a *$\mathcal{P}$-function w.r.t.* $\sqsubseteq$ in X if it satisfies the following conditions:

(i) $\varrho(x,y)\ge 0$ for every comparable $x,y\in X$;

(ii) for any sequences $\{x_n\}_{n=1}^{+\infty}$, $\{y_n\}_{n=1}^{+\infty}$ in X such that x_n and y_n are comparable at each $n\in\mathbb{N}$, if $\lim_{n\to+\infty}x_n=x$ and $\lim_{n\to+\infty}y_n=y$, then $\lim_{n\to+\infty}\varrho(x_n,y_n)=\varrho(x,y)$;

(iii) for any sequences $\{x_n\}_{n=1}^{+\infty}$, $\{y_n\}_{n=1}^{+\infty}$ in X such that x_n and y_n are comparable at each $n\in\mathbb{N}$, if $\lim_{n\to+\infty}\varrho(x_n,y_n)=0$, then $\lim_{n\to+\infty}d(x_n,y_n)=0$.

If, in addition, the following condition is also satisfied:

(A) for any sequences $\{x_n\}_{n=1}^{+\infty}$, $\{y_n\}_{n=1}^{+\infty}$ in X such that x_n and y_n are comparable at each $n\in\mathbb{N}$, if the limit $\lim_{n\to+\infty}d(x_n,y_n)$ exists, then the limit $\lim_{n\to+\infty}\varrho(x_n,y_n)$ also exists,

then ϱ is said to be a *$\mathcal{P}$-function of type (A) w.r.t.* $\sqsubseteq$ in X.

Example 2.2 Let $(X,\sqsubseteq,d)$ be a partially ordered metric space. Suppose that the function $\varphi:[0,+\infty)\to[0,+\infty)$ is defined as in Theorem 1.2. Then $\varphi\circ d$ is a $\mathcal{P}$-function of type (A) w.r.t. $\sqsubseteq$ in X.

Proposition 2.3 *Let $(X,\sqsubseteq,d)$ be a partially ordered metric space and $\varrho : X\times X\to\mathbb{R}$ be a $\mathcal{P}$-function w.r.t. $\sqsubseteq$ in X. If $x,y\in X$ are comparable and $\varrho(x,y)=0$, then $x=y$.*

Proof Let $x, y \in X$ be comparable and $\varrho(x,y) = 0$. Define $\{x_n\}_{n=1}^{+\infty}$ and $\{y_n\}_{n=1}^{+\infty}$ to be two constant sequences in X such that $x_n = x$ and $y_n = y$ for all $n \in \mathbb{N}$. It follows from the definition of a $\mathcal{P}$-function, since x and y are comparable, that $d(x,y) = 0$. That is, $x = y$. □

Corollary 2.4 *Let $(X, \sqsubseteq, d)$ be a totally ordered metric space and $\varrho : X \times X \to \mathbb{R}$ be a $\mathcal{P}$-function w.r.t. $\sqsubseteq$ in X. If $x, y \in X$ and $\varrho(x,y) = 0$, then $x = y$.*

Proof Since X is totally ordered, any $x, y \in X$ are comparable. The rest of the proof is straightforward. □

Example 2.5 Let $X = \mathbb{R}$. Define $d, \varrho : X \times X \to \mathbb{R}$ with $d(x,y) = |x - y|$ and $\varrho(x,y) = 1 + |x - y|$. If X is endowed with a usual ordering $\leq$, then $(X, \leq, d)$ is a totally ordered metric space with ϱ as a $\mathcal{P}$-function of type (A) w.r.t. $\leq$ in X. Note that $\varrho(x,y) \neq 0$ for all $x, y \in X$, even when $x = y$.

This example shows that the converse of Proposition 2.3 and that of Corollary 2.4 are not generally true.

Definition 2.6 Let $(X, \sqsubseteq, d)$ be a partially ordered metric space, a mapping $f : X \to X$ is called a *$\mathcal{P}$-contraction w.r.t.* $\sqsubseteq$ if there exists a $\mathcal{P}$-function $\varrho : X \times X \to \mathbb{R}$ w.r.t. $\sqsubseteq$ in X such that

$$d(fx, fy) \leq d(x,y) - \varrho(x,y) \tag{2.1}$$

for any comparable $x, y \in X$. Naturally, if there exists a $\mathcal{P}$-function of type (A) w.r.t. $\sqsubseteq$ in X such that the inequality (2.1) holds for any comparable $x, y \in X$, then f is said to be a *$\mathcal{P}$-contraction of type (A) w.r.t.* $\sqsubseteq$.

Remark 2.7 From Example 2.2, it follows that in partially ordered metric spaces, a weak contraction is also a $\mathcal{P}$-contraction.

3 Fixed point results

3.1 Fixed point theorems for monotonic mappings

Theorem 3.1 *Let $(X, \sqsubseteq, d)$ be a complete partially ordered metric space and $f : X \to X$ be a continuous and nondecreasing $\mathcal{P}$-contraction of type (A) w.r.t. $\sqsubseteq$. If there exists $x_0 \in X$ with $x_0 \sqsubseteq fx_0$, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a fixed point of f in X.*

Proof For the existence of the fixed point, we choose $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$. If $fx_0 = x_0$, then the proof is finished. Suppose that $fx_0 \neq x_0$. We define a sequence $\{x_n\}_{n=1}^{+\infty}$ such that $x_n = f^n x_0$. Since $x_0 \sqsubseteq fx_0$ and f is nondecreasing w.r.t. $\sqsubseteq$, we obtain

$$x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq x_{n+1} \sqsubseteq \cdots.$$

If there exists $n_0 \in \mathbb{N}$ such that $\varrho(x_{n_0}, x_{n_0+1}) = d(x_{n_0}, x_{n_0+1})$, then by the notion of $\mathcal{P}$-contractivity, the proof is finished. Therefore, we assume that $\varrho(x_n, x_{n+1}) < d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Also, assume that $\varrho(x_n, x_{n+1}) \neq 0$ for all $n \in \mathbb{N}$. Otherwise, we can find $n_0 \in \mathbb{N}$ with $x_{n_0} = x_{n_0+1}$, that is, $x_{n_0} = fx_{n_0}$, and the proof is finished. Hence, we consider only the case where $0 < \varrho(x_n, x_{n+1}) < d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$.

Since $x_n \sqsubseteq x_{n+1}$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(fx_{n-1}, fx_n) \\ &\leq d(x_{n-1}, x_n) - \varrho(x_{n-1}, x_n) \\ &\leq d(x_{n-1}, x_n) \end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, we have $\{d(x_n, x_{n+1})\}_{n=1}^{+\infty}$ nonincreasing. Since $\{d(x_n, x_{n+1})\}_{n=1}^{+\infty}$ is bounded, there exists $l \geq 0$ such that $\lim_{n\to+\infty} d(x_n, x_{n+1}) = l$. Thus, there exists $q \geq 0$ such that $\lim_{n\to+\infty} \varrho(x_n, x_{n+1}) = q$.

Assume that $l > 0$. Then, by the $\mathcal{P}$-contractivity of f, we have

$$l \leq l - q.$$

Hence, $q = 0$, which implies that $l = 0$, a contradiction. Therefore, we have

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0. \tag{3.1}$$

Now we show that $\{x_n\}_{n=1}^{+\infty}$ is a Cauchy sequence in X. Assume the contrary. Then there exists $\epsilon_0 > 0$ for which we can define subsequences $\{x_{m_k}\}_{k=1}^{+\infty}$ and $\{x_{n_k}\}_{k=1}^{+\infty}$ of $\{x_n\}_{n=1}^{+\infty}$ such that n_k is minimal in the sense that $n_k > m_k > k$ and $d(x_{m_k}, x_{n_k}) \geq \epsilon_0$. Therefore, $d(x_{m_k}, x_{n_k-1}) < \epsilon_0$. Observe that

$$\begin{aligned} \epsilon_0 &\leq d(x_{m_k}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) \\ &< \epsilon_0 + d(x_{n_k-1}, x_{n_k}). \end{aligned}$$

Letting $k \to +\infty$, we obtain $\epsilon_0 \leq \lim_{k\to+\infty} d(x_{m_k}, x_{n_k}) \leq \epsilon_0$ and so

$$\lim_{k\to+\infty} d(x_{m_k}, x_{n_k}) = \epsilon_0. \tag{3.2}$$

By the two following inequalities:

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k})$$

and

$$d(x_{m_k-1}, x_{n_k-1}) \leq d(x_{m_k-1}, x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k-1}),$$

we can apply (3.1) and (3.2) to obtain

$$\lim_{k\to+\infty} d(x_{m_k-1}, x_{n_k-1}) = \epsilon_0. \tag{3.3}$$

Furthermore, we deduce that the limit $\lim_{k\to+\infty} \varrho(x_{m_k-1}, x_{n_k-1})$ also exists. Now, by the $\mathcal{P}$-contractivity, we have

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k-1}, x_{n_k-1}) - \varrho(x_{m_k-1}, x_{n_k-1}).$$

From (3.2) and (3.3), we may find that

$$0 \leq -\lim_{k\to+\infty} \varrho(x_{m_k-1}, x_{n_k-1}),$$

which further implies that $\lim_{k\to+\infty} \varrho(x_{m_k-1}, x_{n_k-1}) = 0$. Notice that $x_{m_k-1} \sqsubseteq x_{n_k-1}$ at each $k \in \mathbb{N}$. Consequently, we obtain that $\lim_{k\to+\infty} d(x_{m_k-1}, x_{n_k-1}) = 0$, which is a contradiction. So, $\{x_n\}_{n=1}^{+\infty}$ is a Cauchy sequence. Since X is complete, there exists x^* such that $x_n = f^n x_0 \to x^*$ as $n \to +\infty$. Finally, the continuity of f and $ff^n x_0 = f^{n+1}x_0 \to x^*$ imply that $fx^* = x^*$. Therefore, x^* is a fixed point of f. □

Remark 3.2 In the setting of Remark 2.7, Theorem 3.1 reduces to Theorem 1.2 of [17].

Next, we drop the continuity of f in the Theorem 3.1, and find out that we can still guarantee a fixed point if we strengthen the condition of a partially ordered set to a sequentially ordered set.

Theorem 3.3 *Let $(X, \sqsubseteq, d)$ be a complete sequentially ordered metric space and $f : X \to X$ be a nondecreasing $\mathcal{P}$-contraction of type (A) w.r.t. $\sqsubseteq$. If there exists $x_0 \in X$ with $x_0 \sqsubseteq fx_0$, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a fixed point of f in X.*

Proof If we take $x_n = f^n x_0$ in the proof of Theorem 3.1, then we conclude that $\{x_n\}_{n=1}^{+\infty}$ converges to a point x^* in X.

Next, we prove that x^* is a fixed point of f in X. Indeed, suppose that x^* is not a fixed point of f, *i.e.*, $d(x^*, fx^*) \neq 0$. Since x^* is comparable with x_n for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x^*, fx^*) &\leq d(x^*, fx_n) + d(fx^*, fx_n) \\ &\leq d(x^*, fx_n) + d(x^*, x_n) - \varrho(x^*, x_n) \\ &\leq d(x^*, fx_n) + d(x^*, x_n) \\ &= d(x^*, x_{n+1}) + d(x^*, x_n) \end{aligned}$$

for all $n \in \mathbb{N}$. By the definition of a convergent sequence, we have, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x^*) < \frac{\epsilon}{2}$ for all $n \in \mathbb{N}$ with $n \geq N$. Therefore, we have

$$\begin{aligned} d(x^*, fx^*) &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &< \epsilon. \end{aligned}$$

As easily seen, $d(x^*, fx^*)$ is less than any nonnegative real number, so $d(x^*, fx^*) = 0$, which is a contradiction. Hence, x^* is a fixed point of f. □

Corollary 3.4 *Let $(X, \sqsubseteq, d)$ be a complete totally ordered metric space and $f : X \to X$ be a nondecreasing $\mathcal{P}$-contraction of type (A) w.r.t. $\sqsubseteq$. If there exists $x_0 \in X$ with $x_0 \sqsubseteq fx_0$, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a unique fixed point of f in X.*

Proof Take $x_n = f^n x_0$ as in the proof of Theorem 3.1. Since the total ordering implies the partial ordering, we conclude that $\{x_n\}_{n=1}^{+\infty}$ converges to a fixed point.

Next, we show that the fixed point of f is unique. Assume that u and v are two distinct fixed points of f, *i.e.*, $d(u,v) \neq 0$. Since X is totally ordered, u and v are comparable. Thus, we have

$$\begin{aligned} d(u,v) &= d(fu,fv) \\ &\leq d(u,v) - \varrho(u,v), \end{aligned} \tag{3.4}$$

which is a contradiction. Therefore, $u = v$ and the fixed point of f is unique. □

We can still guarantee the uniqueness of a fixed point by weakening the total ordering condition as stated and proved in the next theorem.

Theorem 3.5 *Let $(X,\sqsubseteq,d)$ be a complete partially ordered metric space and $f : X \to X$ be a continuous and nondecreasing $\mathcal{P}$-contraction of type (A) w.r.t. $\sqsubseteq$. Suppose that for each $x,y \in X$, there exists $w \in X$ which is comparable to both x and y. If there exists $x_0 \in X$ with $x_0 \sqsubseteq fx_0$, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a unique fixed point of f in X.*

Proof If we take $x_n = f^n x_0$ in the proof of Theorem 3.1, then we conclude that x_n converges to a fixed point of f in X.

Next, we show that the fixed point of f is unique. Assume that u and v are two distinct fixed points of f, *i.e.*, $d(u,v) \neq 0$. Since $u,v \in X$, there exists $w \in X$ such that w is comparable to both u and v. We will prove this part by showing that the sequence $\{w_n\}_{n=1}^{+\infty}$ given by $w_n = f^n w$ converges to both u and v. Therefore, we have

$$\begin{aligned} d\bigl(u,f^n w\bigr) &\leq d\bigl(u,f^{n-1}w\bigr) - \varrho\bigl(u,f^{n-1}w\bigr) \\ &\leq d\bigl(u,f^{n-1}w\bigr). \end{aligned} \tag{3.5}$$

If we define a sequence $y_n = d(u,f^n w)$ and $z_n = \varrho(u,f^n w)$, we may obtain from (3.5) that $\{y_n\}_{n=1}^{+\infty}$ is nonincreasing and there exist $l,q \geq 0$ such that $\lim_{n\to+\infty} y_n = l$ and $\lim_{n\to+\infty} z_n = q$.

Assume that $l > 0$. Then, by the $\mathcal{P}$-contractivity of f, we have

$$l \leq l - q,$$

which is a contradiction. Hence, $\lim_{n\to+\infty} y_n = 0$. In the same way, we can also show that $\lim_{n\to+\infty} d(v,f^n w) = 0$. That is, $\{w_n\}_{n=1}^{+\infty}$ converges to both u and v. Since the limit of a convergent sequence in a metric space is unique, we conclude that $u = v$. Hence, this yields the uniqueness of the fixed point. □

Theorem 3.6 *Let $(X,\sqsubseteq,d)$ be a complete sequentially ordered metric space and $f : X \to X$ be a nondecreasing $\mathcal{P}$-contraction of type (A) w.r.t. $\sqsubseteq$. Suppose that for each $x,y \in X$, there exists $w \in X$ which is comparable to both x and y. If there exists $x_0 \in X$ with $x_0 \sqsubseteq fx_0$, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a unique fixed point of f in X.*

Proof If we take $x_n = f^n x_0$ in the proof of Theorem 3.1, then we conclude that x_n converges to a fixed point of f in X. The rest of the proof is similar to the proof of Theorem 3.5. □

Remark 3.7 In parallel with the study of Theorems 3.1, 3.3, 3.5 and 3.6, we can also prove in the same way that if the mapping f is nonincreasing, the above theorems still hold. However, we will omit the result for nonincreasing mappings.

3.2 Fixed point theorems for mappings with the lack of monotonicity

In this section, we drop the monotonicity conditions of f and find out that we can still apply our results to confirm the existence and uniqueness of a fixed point of f.

Theorem 3.8 *Let $(X, \sqsubseteq, d)$ be a complete partially ordered metric space and $f : X \to X$ be a continuous $\mathcal{P}$-contraction of type (A) w.r.t. $\sqsubseteq$ such that the comparability of $x, y \in X$ implies the comparability of $fx, fy \in fX$. If there exists $x_0 \in X$ such that x_0 and fx_0 are comparable, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a fixed point of f in X.*

Proof For the existence of the fixed point, we choose $x_0 \in X$ such that x_0 and fx_0 are comparable. If $fx_0 = x_0$, then the proof is finished. Suppose that $fx_0 \neq x_0$. We define a sequence $\{x_n\}_{n=1}^{+\infty}$ such that $x_n = f^n x_0$. Since x_0 and fx_0 are comparable, we have x_n and x_{n+1} comparable for all $n \in \mathbb{N}$.

If there exists $n_0 \in \mathbb{N}$ such that $\varrho(x_{n_0}, x_{n_0+1}) = d(x_{n_0}, x_{n_0+1})$, then by the notion of $\mathcal{P}$-contractivity, the proof is finished. Therefore, we assume that $\varrho(x_n, x_{n+1}) < d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Also, assume that $\varrho(x_n, x_{n+1}) \neq 0$ for all $n \in \mathbb{N}$. Otherwise, we can find $n_0 \in \mathbb{N}$ with $x_{n_0} = x_{n_0+1}$, that is, $x_{n_0} = fx_{n_0}$, and the proof is finished. Hence, we consider only the case where $0 < \varrho(x_n, x_{n+1}) < d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$.

Since x_n and x_{n+1} are comparable for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(fx_{n-1}, fx_n) \\ &\leq d(x_{n-1}, x_n) - \varrho(x_{n-1}, x_n) \\ &\leq d(x_{n-1}, x_n) \end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, we have $\{d(x_n, x_{n+1})\}_{n=1}^{+\infty}$ is nonincreasing. Since $\{d(x_n, x_{n+1})\}_{n=1}^{+\infty}$ is bounded, there exists $l \geq 0$ such that $\lim_{n\to+\infty} d(x_n, x_{n+1}) = l$. Thus, there exists $q \geq 0$ such that $\lim_{n\to+\infty} \varrho(x_n, x_{n+1}) = q$.

Assume that $l > 0$. Then, by the $\mathcal{P}$-contractivity of f, we have

$$l \leq l - q.$$

Hence, $q = 0$, which implies that $l = 0$, a contradiction. Hence, $\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0$.

Now we show that $\{x_n\}_{n=1}^{+\infty}$ is a Cauchy sequence in X. Assume the contrary. Then there exists $\epsilon_0 > 0$ for which we can define subsequences $\{x_{m_k}\}_{k=1}^{+\infty}$ and $\{x_{n_k}\}_{k=1}^{+\infty}$ of $\{x_n\}_{n=1}^{+\infty}$ such that n_k is minimal in the sense that $n_k > m_k > k$ and $d(x_{m_k}, x_{n_k}) \geq \epsilon_0$. Therefore, $d(x_{m_k}, x_{n_k-1}) < \epsilon_0$. Observe that

$$\begin{aligned} \epsilon_0 &\leq d(x_{m_k}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) \\ &< \epsilon_0 + d(x_{n_k-1}, x_{n_k}). \end{aligned}$$

Letting $k \to +\infty$, we obtain $\epsilon_0 \leq \lim_{k\to+\infty} d(x_{m_k}, x_{n_k}) \leq \epsilon_0$ and so

$$\lim_{k\to+\infty} d(x_{m_k}, x_{n_k}) = \epsilon_0. \tag{3.6}$$

By the two following inequalities:

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k})$$

and

$$d(x_{m_k-1}, x_{n_k-1}) \leq d(x_{m_k-1}, x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k-1}),$$

we can apply the fact that $\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0$ and (3.6) to obtain

$$\lim_{k\to+\infty} d(x_{m_k-1}, x_{n_k-1}) = \epsilon_0. \tag{3.7}$$

Furthermore, we deduce that the limit $\lim_{k\to+\infty} \varrho(x_{m_k-1}, x_{n_k-1})$ also exists. Now, by the $\mathcal{P}$-contractivity, we have

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k-1}, x_{n_k-1}) - \varrho(x_{m_k-1}, x_{n_k-1}).$$

From (3.6) and (3.7), we may find that

$$0 \leq -\lim_{k\to+\infty} \varrho(x_{m_k-1}, x_{n_k-1}),$$

which further implies that $\lim_{k\to+\infty} \varrho(x_{m_k-1}, x_{n_k-1}) = 0$. Notice that $x_{m_k-1} \sqsubseteq x_{n_k-1}$ at each $k \in \mathbb{N}$. Consequently, we obtain that $\lim_{k\to+\infty} d(x_{m_k-1}, x_{n_k-1}) = 0$, which is a contradiction. So, $\{x_n\}_{n=1}^{+\infty}$ is a Cauchy sequence. Since X is complete, there exists x^* such that $x_n = f^n x_0 \to x^*$ as $n \to +\infty$. Finally, the continuity of f and $ff^n x_0 = f^{n+1} x_0 \to x^*$ imply that $fx^* = x^*$. Therefore, x^* is a fixed point of f. □

Further results can be proved using the same plots as those of the earlier theorems in this paper, so we omit them.

Theorem 3.9 *Let $(X, \sqsubseteq, d)$ be a complete sequentially ordered metric space and $f : X \to X$ be a $\mathcal{P}$-contraction of type (A) w.r.t. $\sqsubseteq$ such that the comparability of $x, y \in X$ implies the comparability of $fx, fy \in fX$. If there exists $x_0 \in X$ such that x_0 and fx_0 are comparable, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a fixed point of f in X.*

Corollary 3.10 *Let $(X, \sqsubseteq, d)$ be a complete totally ordered metric space with and $f : X \to X$ be a $\mathcal{P}$-contraction of type (A) w.r.t. $\sqsubseteq$ such that the comparability of $x, y \in X$ implies the comparability of $fx, fy \in fX$. If there exists $x_0 \in X$ such that x_0 and fx_0 are comparable, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a unique fixed point of f in X.*

Theorem 3.11 *Let $(X, \sqsubseteq, d)$ be a complete partially ordered metric space and $f : X \to X$ be a continuous $\mathcal{P}$-contraction of type (A) w.r.t. $\sqsubseteq$ such that the comparability of $x, y \in X$ implies the comparability of $fx, fy \in fX$. Suppose that for each $x, y \in X$, there exists $w \in$*

X which is comparable to both x and y. If there exists $x_0 \in X$ such that x_0 and fx_0 are comparable, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a unique fixed point of f in X.

Theorem 3.12 *Let $(X, \sqsubseteq, d)$ be a complete sequentially ordered metric space and $f : X \to X$ be a $\mathcal{P}$-contraction of type (A) w.r.t. $\sqsubseteq$ such that the comparability of $x, y \in X$ implies the comparability of $fx, fy \in fX$. Suppose that for each $x, y \in X$, there exists $w \in X$ which is comparable to both x and y. If there exists $x_0 \in X$ such that x_0 and fx_0 are comparable, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a unique fixed point of f in X.*

4 Example

We give an example to ensure the applicability of our theorems.

Example 4.1 Let $X = [0,1] \times [0,1]$ and suppose that we write $x = (x_1, x_2)$ and $y = (y_1, y_2)$ for $x, y \in X$.

Define $d, \varrho : X \times X \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 2\max\{x_1 + y_1, x_2 + y_2\} & \text{otherwise} \end{cases}$$

and

$$\varrho(x,y) = \begin{cases} 0 & \text{if } x = y, \\ \max\{x_1, x_2 + y_2\} & \text{otherwise.} \end{cases}$$

Let $\sqsubseteq$ be an ordering in X such that for $x, y \in X$, $x \sqsubseteq y$ if and only if $x_1 = y_1$ and $x_2 \leq y_2$. Then $(X, \sqsubseteq, d)$ is a partially ordered metric space with ϱ as a $\mathcal{P}$-function of type (A) w.r.t. $\sqsubseteq$ in X.

Now, let f be a self mapping on X defined by $fx = f((x_1, x_2)) = (0, \frac{x_2^2}{2})$ for all $x \in X$. It is obvious that f is continuous and nondecreasing w.r.t. $\sqsubseteq$.

Let $x, y \in X$ be comparable w.r.t. $\sqsubseteq$. If $x = y$, then they clearly satisfy the inequality (2.1). On the other hand, if $x \neq y$, we have

$$\begin{aligned} d(fx, fy) &= d\bigl(f\bigl((x_1, x_2)\bigr), f\bigl((y_1, y_2)\bigr)\bigr) \\ &= d\left(\left(0, \frac{x_2^2}{2}\right), \left(0, \frac{y_2^2}{2}\right)\right) \\ &= 2\max\left\{0, \frac{x_2^2}{2} + \frac{y_2^2}{2}\right\} \\ &= x_2^2 + y_2^2 \\ &\leq x_2 + y_2 \\ &\leq \max\{2x_1, x_2 + y_2\} \\ &= 2\max\{2x_1, x_2 + y_2\} - \max\{2x_1, x_2 + y_2\} \\ &\leq 2\max\{2x_1, x_2 + y_2\} - \max\{x_1, x_2 + y_2\} \\ &= d(x,y) - \varrho(x,y). \end{aligned}$$

Therefore, the inequality (2.1) is satisfied for every comparable $x, y \in X$. So, f is a continuous and nondecreasing $\mathcal{P}$-contraction of type (A) w.r.t. $\sqsubseteq$. Let $x_0 = (0,0)$, so we have $x_0 \sqsubseteq fx_0$. Now, applying Theorem 3.1, we conclude that f has a fixed point in X which is the point $(0,0)$.

5 Conclusion

It is undeniable that Rhoades's weak contraction is one of the earliest and the most important extensions of the contraction principle. The results in this paper give a new direction to expanding the framework of contractive type mappings in metric spaces. Still, there is a question to be raised from this paper onwards.

Question Are our results still true for any $\mathcal{P}$-contractions (not necessarily of type (A))?

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed significantly in writing this paper. They have also read and approved the final manuscript.

Acknowledgements
This work was partially supported by the Higher Education Research Promotion and National Research University Project of Thailand, Office of the Higher Education Commission (NRU-CSEC No. 55000613). The second author would like to thank the Research Professional Development Project under the Science Achievement Scholarship of Thailand (SAST). Also, the third author was supported by the Commission on Higher Education, the Thailand Research Fund and the King Mongkut's University of Technology Thonburi (Grant No. MRG5580213).

Received: 19 March 2012 Accepted: 16 November 2012 Published: 5 December 2012

References

1. Aydi, H, Abbas, M, Sintunavarat, W, Kumam, P: Tripled fixed point of W-compatible mappings in abstract metric spaces. Fixed Point Theory Appl. **2012**, 134 (2012)
2. Aydi, H, Vetro, C, Sintunavarat, W, Kumam, P: Coincidence and fixed points for contractions and cyclical contractions in partial metric spaces. Fixed Point Theory Appl. **2012**, 124 (2012)
3. Ćirić, L: A generalization of Banach's contraction principle. Proc. Am. Math. Soc. **45**, 267-273 (1974)
4. Ćirić, L: A new fixed-point theorem for contractive mappings. Publ. Inst. Math. (Belgr.) **30**, 25-27 (1981)
5. Ćirić, L: Solving the Banach fixed point principle for nonlinear contractions in probabilistic metric spaces. Nonlinear Anal., Theory Methods Appl. **72**(3-4), 2009-2018 (2010)
6. Chaipunya, P, Cho, YJ, Kumam, P: Geraghty-type theorems in modular metric spaces with an application to partial differential equation. Adv. Differ. Equ. **2012**, 83 (2012)
7. Chaipunya, P, Mongkolkeha, C, Sintunavarat, W, Kumam, P: Fixed-point theorems for multivalued mappings in modular metric spaces. Abstr. Appl. Anal. **2012**, Article ID 503504 (2012)
8. Chaipunya, P, Mongkolkeha, C, Sintunavarat, W, Kumam, P: Erratum to "Fixed-point theorems for multivalued mappings in modular metric spaces". Abstr. Appl. Anal. **2012**, Article ID 241919 (2012)
9. Mongkolkeha, C, Sintunavarat, W, Kumam, P: Fixed point theorems for contraction mappings in modular metric spaces. Fixed Point Theory Appl. **2011**, 93 (2011)
10. Mongkolkeha, C, Sintunavarat, W, Kumam, P: Correction: Fixed point theorems for contraction mappings in modular metric spaces. Fixed Point Theory Appl. 2011, 2011:93. Fixed Point Theory Appl. **2012**, 103 (2012)
11. Sintunavarat, W, Cho, YJ, Kumam, P: Coupled fixed point theorems for contraction mapping induced by cone ball-metric in partially ordered spaces. Fixed Point Theory Appl. **2012**, 128 (2012)
12. Sintunavarat, W, Cho, YJ, Kumam, P: Coupled fixed point theorems for weak contraction mappings under F-invariant set. Abstr. Appl. Anal. **2012**, Article ID 324874 (2012)
13. Sintunavarat, W, Kumam, P: Weak condition for generalized multi-valued (f, α, β)-weak contraction mappings. Appl. Math. Lett. **24**(4), 460-465 (2011)
14. Suzuki, T: A generalized Banach contraction principle that characterizes metric completeness. Proc. Am. Math. Soc. **136**(5), 1861-1869 (2008)
15. Alber, YI, Guerre-Delabriere, S: Principles of weakly contractive maps in Hilbert spaces. In: Gohberg, I, Lyubich, Y (eds.) New Results in Operator Theory. Advances and Appl., vol. 98, pp. 7-22. Birkhäuser, Basel (1997)
16. Rhoades, BE: Some theorems on weakly contractive maps. Nonlinear Anal. **47**, 2683-2693 (2001)
17. Harjani, J, Sadarangani, K: Fixed point theorems for weakly contractive mappings in partially ordered sets. Nonlinear Anal. **71**, 3403-3410 (2009)

doi:10.1186/1687-1812-2012-219
Cite this article as: Chaipunya et al.: **On $\mathcal{P}$-contractions in ordered metric spaces.** *Fixed Point Theory and Applications* 2012 **2012**:219.